

algebras of pseudodifferential operators

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Algebras of Pseudodifferential Operators: A Comprehensive Guide

In modern analysis, the study of differential equations on manifolds has evolved far beyond classical differential operators. Pseudodifferential operators (PDOs) generalize differential operators by allowing symbols that capture nonlocal behavior, making them indispensable tools in partial differential equations, microlocal analysis, and global analysis. When we organize PDOs into algebraic structures—known as **algebras of pseudodifferential operators**—we unlock powerful techniques for composition, inversion, and spectral theory. This article offers a deep dive into the theory, structure, and applications of these algebras, aiming to illuminate their significance for both researchers and advanced students.

1. Foundations: What Are Pseudodifferential Operators?

1.1 Definition via Symbols

A pseudodifferential operator on a smooth manifold (M) is an operator (P) acting on smooth functions (or sections of a vector bundle) that, in local coordinates, can be represented as

$$P(u)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} p(x, \xi) u(y) dy d\xi,$$

where $(p(x, \xi))$ is called the **symbol** of (P) . The symbol encodes how the operator behaves in both position (x) and frequency (ξ) . Classical symbols satisfy asymptotic expansions in powers of $(|\xi|)$, while more general symbols may belong to Hörmander's $(S^m_{\rho, \delta})$ classes.

1.2 Relation to Differential Operators

Every differential operator is a pseudodifferential operator with a polynomial symbol. Conversely, many nonlocal phenomena—such as fractional Laplacians or Fourier multipliers—are captured by PDOs with symbols that are not polynomials. This broader framework allows us to treat operators that are not strictly local, thereby expanding the toolbox for solving PDEs.

1.3 Basic Properties

- Linearity: PDOs are linear operators.
- Composition: The composition of two PDOs is again a PDO, with a symbol given by a nontrivial asymptotic expansion.
- Adjoints: The adjoint of a PDO is a PDO whose symbol is the complex conjugate of the original symbol, modulo lower-order terms.
- Ellipticity: A PDO is elliptic if its principal symbol is invertible away from the zero section; elliptic operators have parametrices within the pseudodifferential calculus.

2. Symbol Calculus: The Engine Behind the Algebra

2.1 Composition Formula

Given two PDOs (P) and (Q) with symbols $(p(x, \xi))$ and $(q(x, \xi))$, their composition $(P \circ Q)$ has a symbol

$\mathcal{S}'(x, \hbar)$ given by the asymptotic series

$$\mathcal{S}'(x, \hbar) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_x^\alpha p(x, \hbar) \hbar^\alpha + \sum_{\alpha} \frac{1}{\alpha!} \partial_x^\alpha q(x, \hbar) \hbar^\alpha.$$

This formula demonstrates why the set of PDOs is closed under composition: the series converges in the appropriate symbol class, ensuring that $\mathcal{S}'(x, \hbar)$ remains a valid symbol.

2.2 Asymptotic Expansions

Symbols often admit an asymptotic expansion of the form

$$p(x, \hbar) \sim \sum_{j=0}^{\infty} p_{-j}(x, \hbar) \hbar^j,$$

where each p_{-j} is homogeneous of degree $-(m-j)$ in $(\hbar)$. This decomposition separates the principal part (highest-order term) from lower-order corrections, which is essential for defining ellipticity and constructing parametrices.

2.3 Moyal Product and Star-Products

In deformation quantization, the composition of PDOs can be expressed via the Moyal product, a noncommutative star-product on the space of symbols. This perspective links PDO algebras to quantum mechanics and symplectic geometry, offering a richer algebraic structure.

3. The Structure of Algebras of Pseudodifferential Operators

3.1 Definition of a PDO Algebra

A **pseudodifferential algebra** is a set $\mathcal{A}$ of PDOs closed under addition, scalar multiplication, composition, and taking adjoints. Typically, $\mathcal{A}$ is a subalgebra of the full algebra of all PDOs on a manifold, tailored to specific properties such as support conditions or order bounds.

3.2 Classical vs. Non-Classical Algebras

- **Classical PDO Algebra:** Contains operators with symbols that admit a full asymptotic expansion in homogeneous terms. This algebra is stable under composition and adjoints.
- **Hörmander Algebra:** Consists of symbols in $S^m_{\rho, \delta}$ classes, allowing for more general behavior (e.g., limited regularity or growth).
- **SG (Shubin–Gindikin) Algebra:** Handles symbols with separate behavior in spatial and frequency variables, useful for scattering theory.

3.3 Filtration by Order

Each algebra is naturally filtered by the order of its elements. For a given m , define

$$\Psi^m(M) = \{ P \mid \text{order}(P) \leq m \}.$$

The associated graded algebra captures the principal symbols, leading to a short exact sequence

$$0 \rightarrow \Psi^{m-1}(M) \rightarrow \Psi^m(M) \xrightarrow{\sigma_m} S^m(M) \rightarrow 0,$$

where σ_m is the principal symbol map. This filtration underpins many analytical results, such as the existence of parametrices for elliptic operators.

4. Elliptic Pseudodifferential Operators and Parametrices

4.1 Ellipticity Condition

An operator $P \in \Psi^m(M)$ is called **elliptic** if its principal symbol $\sigma_m(P)(x, \xi)$ is invertible for all $(x, \xi) \neq 0$. Ellipticity guarantees that P behaves like an isomorphism modulo smoothing operators.

4.2 Construction of Parametrices

For an elliptic operator P , one can construct a parametrix $Q \in \Psi^{-m}(M)$ such that

$$P \circ Q = I + R, \quad Q \circ P = I + R',$$

where R, R' are smoothing operators (operators with infinitely smoothing symbols). This parametrix construction relies on the symbol calculus and the ability to invert the principal symbol asymptotically.

4.3 Fredholm Property

On a compact manifold, an elliptic PDO defines a Fredholm operator between suitable Sobolev spaces. The index of this operator depends only on topological data, leading to deep connections with K-theory and index theory.

5. Index Theory and Global Analysis

5.1 Atiyah–Singer Index Theorem

The celebrated Atiyah–Singer Index Theorem links the analytic index of an elliptic PDO to topological invariants of the underlying manifold. For a Dirac-type operator D , the theorem states

$$\operatorname{ind}(D) = \langle \hat{A}(M) \cup \operatorname{ch}(E), [M] \rangle,$$

where $\hat{A}(M)$ is the A-roof genus and $\operatorname{ch}(E)$ is the Chern character of a vector bundle E .

5.2 Spectral Asymptotics

PDO algebras provide a framework for studying the spectrum of elliptic operators. Weyl's law, for example, describes the asymptotic distribution of eigenvalues of a positive elliptic operator in terms of the volume of the cosphere bundle.

5.3 Heat Kernel Asymptotics

Through the functional calculus of PDOs, one can derive the short-time asymptotic expansion of the heat kernel e^{-tP} . Coefficients in this expansion are local invariants, connecting analysis with geometry.

6. Applications Across Mathematics and Physics

6.1 Partial Differential Equations

Pseudodifferential algebras are central to the theory of linear PDEs, enabling the construction of fundamental solutions, regularity results, and propagation of singularities. The microlocal perspective reveals how singularities travel along bicharacteristics.

6.2 Scattering Theory

In scattering problems on noncompact manifolds, the SG algebra of PDOs captures the behavior of waves at infinity. This algebra accommodates variable coefficients that decay at infinity, facilitating the analysis of scattering matrices.

6.3 Quantum Field Theory

Renormalization in quantum field theory often involves manipulating nonlocal operators. The algebraic structure of PDOs, especially their symbolic calculus, provides a rigorous foundation for regularization and renormalization schemes.

6.4 Noncommutative Geometry

Connes' noncommutative geometry uses spectral triples $(\mathcal{A}, \mathcal{H}, D)$ where (D) is a Dirac-type PDO. The algebra $\mathcal{A}$ of functions is extended to include pseudodifferential operators, allowing the study of spaces that lack a classical pointwise structure.

7. Advanced Topics and Recent Developments

7.1 Microlocal Analysis on Manifolds with Boundary

When manifolds have boundaries, the calculus of PDOs must be adapted. Boutet de Monvel's algebra incorporates boundary conditions, enabling the treatment of boundary value problems.

7.2 Edge and Corner Algebras

For manifolds with singularities (edges, corners), specialized pseudodifferential algebras capture the behavior near the singularities. These algebras are crucial in the analysis of elliptic problems on stratified spaces.

7.3 Global Symbolic Calculus on Lie Groups

On Lie groups, PDO algebras can be defined using group representations. The global symbolic calculus on compact Lie groups leads to explicit

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